

# Dipole perturbations of the Reissner-Nordström solution: II. The axial case

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We study the linear metric perturbations of the Reissner-Nordström solution for the case of axial perturbation modes. We find that the well-known perturbative analysis fails for the case of dipole ( $l = 1$ ) perturbations, although valid for higher multipoles. We define new radial functions, with which the perturbation formalism is generalized to all multipole orders, including the dipole. We then complete the solution by constructing the perturbed metric and Maxwell tensors.

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## I. INTRODUCTION

The linear gravitational and electromagnetic perturbations of the Reissner-Nordström solution have been analyzed by Moncrief [1], Zerilli [2], Chandrasekhar [3], and Chandrasekhar and Xanthopoulos [4]. (For a summary of these works see Refs. [5,6]. These perturbations were considered also in [7–10].) In the perturbative approach the deviations from the exact Reissner-Nordström solution are considered as (small) perturbations over the Reissner-Nordström background. Due to the coupling of the electromagnetic and the gravitational fields in the background, any gravitational perturbation leads in general via the field equations to an electromagnetic perturbation, and vice versa. Despite this complication it turns out that the equations describing the evolution of the perturbations can be decoupled (for each multipole order and for each parity) to a pair of independent one-dimensional wave equations (namely, the Moncrief-Zerilli equations), each of which is made of an electromagnetic component and a gravitational component.

In a recent paper [11] (hereafter Paper I) we analyzed the polar modes of the perturbations, and showed that the perturbative formalism of Ref. [5] fails in the case of dipole perturbations (namely, the  $l = 1$  modes in a multipole expansion). We generalized the perturbative analysis to include also the dipole modes by defining new radial functions, and completed the solution by giving the metric perturbations and the perturbations of the Maxwell tensor in terms of the perturbing fields. In this sequel we treat the axial perturbations of the Reissner-Nordström solution. (For a definition of the parities see, e.g., Ref. [5].) As we shall see in what follows, also in the axial case the formalism of Ref. [5] fails, and for similar reasons to the failure in the polar case discussed in Ref. [11]. In this paper we shall again define new radial functions, with which one can formulate a perturbative analysis which is valid for all multipole orders, including the dipole.

For many applications the failure of the perturbative formalism for the dipole modes is immaterial. This is due to the non-radiative character of dipole gravitational modes. However, there are cases where dipole *electromagnetic* waves are important, especially in the Reissner-

Nordström spacetime, because of the coupling of the electromagnetic and gravitational fields. For example, the late-time behavior of the electromagnetic perturbations produced during the collapse is controlled by the dipole modes (which are the least-damped modes) [12–14]. In addition, the dipole perturbations are especially important in the analysis of the (electromagnetic) perturbations of the Cauchy horizon of Reissner-Nordström black holes [15]. One needs, therefore, a valid perturbative formalism for the calculation of the perturbed metric coefficients and the components of the Maxwell tensor.

The organization of this paper is as follows: In section II we discuss our definitions and the notations. In section III we review the perturbative approach we use. In section IV we derive the coupled equations, which we decouple in section V, and in section VI we complete the solution, which is adequate for the treatment of the dipole modes.

## II. DEFINITIONS AND NOTATION

Following the notation and convention of Ref. [5], we write the line element of an unperturbed Reissner-Nordström black hole in the form

$$ds^2 = e^{2\nu} (dx^0)^2 - e^{2\mu_2} (dx^2)^2 - r^2 d\Omega^2, \quad (1)$$

where  $d\Omega^2 = e^{2\psi} (dx^1)^2 + e^{2\mu_3} (dx^3)^2$  is the line element on the unit two-sphere, the metric coefficients are  $e^{2\nu} = e^{-2\mu_2} = (r^2 - 2Mr + Q_*^2)/r^2 \equiv \Delta/r^2$  and  $e^{2\psi} = e^{2\mu_3} \sin^2 \theta = r^2 \sin^2 \theta$ , and the coordinates are  $(x^0 x^1 x^2 x^3) = (t \phi r \theta)$ . Here,  $r$  is the radial Schwarzschild coordinate, defined such that circles of radius  $r$  have circumference  $2\pi r$ , and  $M$  and  $Q_*$  are the mass and electric charge, correspondingly, of the black hole.

It turns out that the metric perturbations of the Reissner-Nordström black hole can be separated to two parities—called polar (or even parity) and axial (or odd parity)—in accordance with their behavior under the transformation  $\phi \rightarrow -\phi$ . The Reissner-Nordström background, being spherically symmetric and static, is characterized by the vanishing of all the axial metric coefficients

[namely, in the background one has  $\omega = q_2 = q_3 = 0$ ; see the definitions for these metric coefficients below in Eq. (2)]. Thus, any axial perturbation will impart rotation on the black hole, and transform it into a (slowly) rotating Kerr-Newman black hole. It can be shown [5] that the metric

$$ds^2 = e^{2\nu} (dx^0)^2 - e^{2\psi} (dx^1 - \omega dx^0 - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2} (dx^2)^2 - e^{2\mu_3} (dx^3)^2. \quad (2)$$

is of sufficient generality for the treatment of all perturbations (polar and axial).

Throughout this paper we shall use the notation of Ref. [5], except when we change its formalism; Then, similarly to the notation of Paper I, we shall add a bar to the symbols of Ref. [5]. The ‘barred’ objects will be defined in such a way that dipole perturbations are treated adequately. This notation facilitates the comparison between our formalism, and the formalism of Ref. [5].

### III. THE PERTURBATIVE FORMALISM

In a similar way to the development of the perturbative equations for the polar modes, the equations governing the axial perturbations are derived by a linearization of the coupled Einstein-Maxwell equations about the Reissner-Nordström background [5]. In particular, after substitution of the expressions for the unperturbed metric coefficients (of the background) in the linearized  $\phi r$  and  $\phi \theta$  components of the Ricci tensor, one obtains the equations

$$\begin{aligned} [r^2 e^{2\nu} (q_{2,3} - q_{3,2}) \sin^3 \theta]_{,3} + r^4 (\omega_{,2} - q_{2,0})_{,0} \sin^3 \theta \\ = 2r^3 e^\nu \sin^2 \theta \delta R_{(1)(2)} \\ = 4Q_* r e^\nu F_{(0)(1)} \sin^2 \theta \end{aligned} \quad (3)$$

and

$$\begin{aligned} [r^2 e^{2\nu} (q_{2,3} - q_{3,2}) \sin^3 \theta]_{,2} - r^2 e^{-2\nu} (\omega_{,3} - q_{3,0})_{,0} \sin^3 \theta \\ = -2r^2 \sin^2 \theta \delta R_{(1)(3)} \\ = 0, \end{aligned} \quad (4)$$

where  $X_{(\alpha)(\beta)}$  is the  $(\alpha)(\beta)$  tetrad component of the tensor  $X$ . Here,  $R$  and  $F$  are the Ricci and Maxwell tensors, respectively, and a comma denotes partial differentiation. Substitution in the linearized components of the Maxwell tensor yields

$$[r e^\nu F_{(0)(1)}]_{,2} + r e^{-\nu} F_{(1)(2),0} = 0, \quad (5)$$

$$r e^\nu [F_{(0)(1)} \sin \theta]_{,3} + r^2 F_{(1)(3),0} \sin \theta = 0, \quad (6)$$

$$\begin{aligned} r e^{-\nu} F_{(0)(1),0} + [r e^\nu F_{(1)(2)}]_{,2} + F_{(1)(3),3} \\ = -Q_* (\omega_2 - q_{2,0}) \sin \theta. \end{aligned} \quad (7)$$

We now introduce the functions

$$B(r, \theta) = F_{(0)(1)} \sin \theta, \quad (8)$$

$$Q(r, \theta) = r^2 e^{2\nu} (q_{2,3} - q_{3,2}) \sin^3 \theta. \quad (9)$$

After substitution of these functions in Eqs. (3) and (4), and using the linearized Maxwell equations [Eqs. (5)–(7)], we obtain the differential equations

$$\frac{1}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial \theta} = -(\omega_{,2} - q_{2,0})_{,0} + \frac{4Q_*}{r^3 \sin^2 \theta} e^\nu B \quad (10)$$

and

$$\frac{\Delta}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial r} = (\omega_{,3} - q_{3,0})_{,0}. \quad (11)$$

We now assume that the perturbations can be analyzed to their normal modes with a time dependence  $e^{i\sigma t}$ . (This is always possible for a linear theory.) Because of this explicit time-dependence of the metric perturbations we did not consider the time dependence in the definitions for the functions  $B(r, \theta)$  and  $Q(r, \theta)$ . Substitution in Eqs. (10) and (11) yields then

$$\frac{1}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial \theta} = -i\sigma \omega_{,2} - \sigma^2 q_2 + \frac{4Q_*}{r^3 \sin^2 \theta} e^\nu B \quad (12)$$

and

$$\frac{\Delta}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial r} = i\sigma \omega_{,3} + \sigma^2 q_3. \quad (13)$$

Differentiating Eq. (12) with respect to  $\theta$  and Eq. (13) with respect to  $r$ , and summing the equations, we find that

$$\begin{aligned} r^4 \frac{\partial}{\partial r} \left( \frac{\Delta}{r^4} \frac{\partial Q}{\partial r} \right) + \sin^3 \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin^3 \theta} \frac{\partial Q}{\partial \theta} \right) + \sigma^2 \frac{r^4}{\Delta} Q \\ = 4Q_* e^\nu r \frac{\partial}{\partial \theta} \left( \frac{B}{\sin^2 \theta} \right) \sin^3 \theta. \end{aligned} \quad (14)$$

We now differentiate Eq. (5) with respect to  $r$ , Eq. (6) with respect to  $\theta$ , and Eq. (7) with respect to  $t$ . Substituting in the latter and using Eq. (8), we find

$$\begin{aligned} [e^{2\nu} (r e^\nu B)_{,2}]_{,2} + \frac{e^\nu}{r} \left( \frac{B_{,3}}{\sin \theta} \right)_{,3} \sin \theta - r e^{-\nu} B_{,0,0} \\ = Q_* (\omega_{,2,0} - q_{2,0,0}) \sin^2 \theta. \end{aligned} \quad (15)$$

Eqs. (14) and (15) govern the axial perturbations.

We next separate the variables in  $B(r, \theta)$  and  $Q(r, \theta)$ . This separation is done in Ref. [5] by the ansatz

$$B(r, \theta) = 3B(r) C_{l+1}^{-1/2}(\cos \theta) \quad (16)$$

and

$$Q(r, \theta) = Q(r) C_{l+2}^{-3/2}(\cos \theta), \quad (17)$$

where  $C_n^\nu$  is the Gegenbauer function of order  $n$  and index  $\nu$  [16]. It turns out that the relations given in Ref. [5] between the Gegenbauer functions and the Legendre functions are correct only for  $l \geq 2$ . (See Eqs. (21) and (22) in Chapter 4 of Ref. [5].) For  $l = 1$ , however, they give erroneous results. The correct expression for the  $n = 3$  and  $\nu = -3/2$  Gegenbauer function is  $C_3^{-3/2}(\cos \theta) = \frac{1}{2}(\cos^3 \theta - 3 \cos \theta)$ , whereas the expressions given in Ref. [5] give (incorrectly) an identically-vanishing expression. For reference, we give here an expression from which the Gegenbauer functions can be calculated directly for all orders and indices [16]:

$$C_n^\nu(\cos \theta) = \sum_{m=0}^n \frac{\Gamma(\nu+m)\Gamma(\nu+n-m)}{m!(n-m)!\Gamma(\nu)^2} \cos(n-2m)\theta.$$

Substituting Eqs. (16) and (17) in Eqs. (14) and (15) we obtain the radial equations

$$\begin{aligned} \Delta \frac{d}{dr} \left( \frac{\Delta}{r^4} \frac{dQ}{dr} \right) - (l-1)(l+2) \frac{\Delta}{r^4} Q + \sigma^2 Q \\ = -\frac{4Q_*}{r^3} (l-1)(l+2) \Delta e^\nu B \end{aligned} \quad (18)$$

and

$$\begin{aligned} \frac{d}{dr} \left[ e^{2\nu} \frac{d}{dr} (r e^\nu B) \right] - l(l+1) \frac{e^\nu}{r} B \\ + \left( \sigma^2 r e^{-\nu} - \frac{4Q_*^2}{r^3} e^\nu \right) B = -Q_* \frac{Q}{r^4}. \end{aligned} \quad (19)$$

#### IV. DERIVATION OF THE COUPLED EQUATIONS

We now define the functions

$$\bar{H}_1^{(-)} = -2r e^\nu B(r) \quad (20)$$

and

$$\bar{H}_2^{(-)} = \frac{1}{r} Q(r). \quad (21)$$

Note, that here we deviated from the formalism of Ref. [5]. The reason for this deviation is as follows: In Eq. (142) of Chapter 5 of Ref. [5] the variables  $\mu$  and  $n$  (do not confuse  $n$  here with the order of the Gegenbauer functions) are defined by  $\mu^2 = 2n = (l-1)(l+2)$ . Then, when the function  $H_1^{(-)}$  is subsequently defined (Eq. (143) of Chapter 5 of Ref. [5]) it is divided by  $\mu$ . However, for dipole modes  $l = 1$ , and consequently  $\mu$  vanishes identically. As  $B$  and  $H_1^{(-)}$  are physically-meaningful functions [they determine the electromagnetic field via Eq. (8) and the Maxwell equations] this is manifestly inappropriate. By keeping this ill-defined functions for the treatment of dipole perturbations, one might encounter

divisions and multiplications of finite physical fields by identically-vanishing expressions, and thus obtain non-sensical results. It should be stressed, however, that for all other modes, namely, for  $l \geq 2$ , the perturbative formalism as presented in Ref. [5] is perfectly correct and valid.

We now change variables to the Regge-Wheeler “tortoise” coordinate  $r_*$  defined by  $d/dr_* = (\Delta/r^2)d/dr$ , and find that  $\bar{H}_1^{(-)}$  and  $\bar{H}_2^{(-)}$  satisfy a pair of coupled second-order differential equations

$$\begin{aligned} \Lambda^2 \bar{H}_1^{(-)} = \frac{\Delta}{r^5} \left\{ \left[ l(l+1)r - 3M + 4\frac{Q_*^2}{r} \right] \bar{H}_1^{(-)} \right. \\ \left. + 3M \bar{H}_1^{(-)} + 2Q_* \bar{H}_2^{(-)} \right\} \end{aligned} \quad (22)$$

and

$$\begin{aligned} \Lambda^2 \bar{H}_2^{(-)} = \frac{\Delta}{r^5} \left\{ \left[ l(l+1)r - 3M + 4\frac{Q_*^2}{r} \right] \bar{H}_2^{(-)} \right. \\ \left. - 3M \bar{H}_2^{(-)} + 2Q_* (l-1)(l+2) \bar{H}_1^{(-)} \right\}. \end{aligned} \quad (23)$$

Here,  $\Lambda^2 = d^2/dr_*^2 + \sigma^2$ . It is important to notice that Eqs. (22) and (23) are already decoupled in the following two cases: First, when  $Q_* = 0$ , namely, when the electric charge of the black hole vanishes and the Reissner-Nordström black hole degenerates to Schwarzschild. In this case there should be just one radiative perturbative mode, described by the Regge-Wheeler equation, which describes the perturbations of Schwarzschild. Indeed, we find that Eq. (23) reduces to the Regge-Wheeler equation in the limit of vanishing electric charge. The other equation apparently describes another radiative mode (as it is a wave-like differential equation). We shall discuss the meaning of the other apparent mode in the next section. Second, when  $l = 1$ , namely, in the case of dipole perturbations. (In fact, in this case Eq. (23) governing  $\bar{H}_2^{(-)}$  is decoupled, while Eq. (22) still couples  $\bar{H}_1^{(-)}$  and  $\bar{H}_2^{(-)}$ . However, in such a case it is very easy to decouple the equations.) In the next section, after we decouple the equations, we shall discuss this case in detail.

#### V. THE DECOUPLING OF THE EQUATIONS

##### A. Decoupling the equations for the dipole case

In a similar way to the decoupling procedure we used in the polar case in Paper I, we first examine the  $l = 1$  case. As already noted in the preceding section, in this case the equation governing the evolution of  $\bar{H}_2^{(-)}$  is already decoupled from  $\bar{H}_1^{(-)}$ . It is easy to verify [by direct substitution in Eqs. (22) and (23)] that the complete decoupling can be obtained by a transformation to the new radial functions  $\bar{Z}_1^{(-)}$  and  $\bar{Z}_2^{(-)}$  defined by

$$\bar{H}_1^{(-)} = \frac{1}{q_1} \left( \bar{Z}_1^{(-)} + 2 \frac{Q_*}{q_1} \bar{Z}_2^{(-)} \right) \quad (24)$$

and

$$\bar{H}_2^{(-)} = -\frac{6M}{q_1^2} \bar{Z}_2^{(-)}. \quad (25)$$

Here,  $q_1$  is an arbitrary parameter. (It will be fixed when we decouple the equations for any  $l$  below.) The factor  $(-6M)$  in Eq. (25) was taken for convenience. We then find the equations for the functions  $\bar{Z}_1^{(-)}$  and  $\bar{Z}_2^{(-)}$  to be

$$\Lambda^2 \bar{Z}_1^{(-)} = \frac{\Delta}{r^5} \left( 2r + 4 \frac{Q_*^2}{r} \right) \bar{Z}_1^{(-)} \quad (26)$$

and

$$\Lambda^2 \bar{Z}_2^{(-)} = \frac{\Delta}{r^5} \left( 2r - 6M + 4 \frac{Q_*^2}{r} \right) \bar{Z}_2^{(-)}. \quad (27)$$

Eq. (27) is just the Regge-Wheeler equation for the case  $l = 1$ , and describes a true physical radiative mode (for the electromagnetic field). Eq. (26), however, does not describe any physical mode, as it is nothing but the evolution equation for the *violation* of the momentum constraint, which propagates hyperbolically on its own (cf. Eq. (68) of Ref. [17] and the discussion therein). If we assume the momentum constraint to be satisfied on the initial spacelike hypersurface (namely, if we take both  $\bar{Z}_1^{(-)}$  and its time derivative to vanish on the initial slice), then  $\bar{Z}_1^{(-)}$  will vanish forever, and consequently Eq. (26) carries no physical information. Consequently, both  $\bar{H}_1^{(-)}$  and  $\bar{H}_2^{(-)}$  are proportional to  $\bar{Z}_2^{(-)}$ , and Eqs. (24) and (25) are diffeomorphic and describe the same physical radiative mode of the electromagnetic field. The reason for the appearance of the equation for the evolution of the violation of the constraint equation is clear. The field equations we use are unconstrained, and therefore one should not be surprised to find non-trivial equations which describe the unphysical sector of the unconstrained theory. The field equations we used are the  $\phi r$  and the  $\phi \theta$  components of the Einstein equations. In order to have a constrained theory, one should include also the  $t \theta$  and  $t \phi$  components, which will insure that unphysical modes will not appear.

Hence, we conclude that in this case there is just one radiative mode, as expected from the non-radiative character of the dipole gravitational field. We add here that in the dipole mode of the polar perturbations (see Paper I) all the physical observables (i.e., the metric coefficients and the components of the Maxwell tensor) are obtained from just one differential equation, even though there appear to be two such equations, again as should be expected from the nature of the dipole modes. As in the axial case we study here, again the unphysical mode is an autonomous evolution of the violation of the momentum constraint. In the axial case one should preclude the

other differential equation for the evolution of  $\bar{Z}_1^{(-)}$ , and thus find that the functions  $\bar{H}_1^{(-)}$  and  $\bar{H}_2^{(-)}$  are diffeomorphic (they are distinguished by just a multiplicative factor).

## B. Decoupling the equation for the general case

We now turn to the decoupling of the equations in the general case (i.e., for any  $l$ ). We shall follow the decoupling procedure of Paper I. We seek functions  $\bar{Z}_1^{(-)}$  and  $\bar{Z}_2^{(-)}$  given by the ansatz

$$\bar{H}_1^{(-)} = \alpha \bar{Z}_1^{(-)} + \beta \bar{Z}_2^{(-)} \quad (28)$$

and

$$\bar{H}_2^{(-)} = \gamma \bar{Z}_1^{(-)} + \delta \bar{Z}_2^{(-)}, \quad (29)$$

where  $\alpha, \beta, \gamma$ , and  $\delta$  are parameters yet to be fixed. Substitution into Eqs. (22) and (23) yields

$$\begin{aligned} \alpha \Lambda^2 \bar{Z}_1^{(-)} + \beta \Lambda^2 \bar{Z}_2^{(-)} &= \frac{\Delta}{r^5} \left[ 2Q_* \gamma + l(l+1)r\alpha + 4 \frac{Q_*^2}{r} \alpha \right] \bar{Z}_1^{(-)} \\ &+ \frac{\Delta}{r^5} \left[ 2Q_* \delta + l(l+1)r\beta + 4 \frac{Q_*^2}{r} \beta \right] \bar{Z}_2^{(-)} \end{aligned} \quad (30)$$

and

$$\begin{aligned} \gamma \Lambda^2 \bar{Z}_1^{(-)} + \delta \Lambda^2 \bar{Z}_2^{(-)} &= \frac{\Delta}{r^5} [2Q_*(l-1)(l+2)\alpha \\ &+ l(l+1)r\gamma - 6M\gamma + 4 \frac{Q_*^2}{r} \gamma] \bar{Z}_1^{(-)} \\ &+ \frac{\Delta}{r^5} [2Q_*(l-1)(l+2)\beta + l(l+1)r\delta \\ &- 6M\delta + 4 \frac{Q_*^2}{r} \delta] \bar{Z}_2^{(-)}. \end{aligned} \quad (31)$$

We now multiply Eq. (30) by  $\gamma$  and Eq. (31) by  $\alpha$ , and subtract the two equations. To have the equations decoupled we require that

$$2Q_*\gamma^2 + 6M\alpha\gamma - 2Q_*(l-1)(l+2)\alpha^2 = 0.$$

This is a quadratic equation in  $\gamma$ , say, whose solution can be determined uniquely, once we recall that  $\gamma$  vanishes for  $l = 1$ . Hence, we find that

$$\gamma = -\frac{1}{2} \frac{\alpha}{Q_*} \left[ 3M - \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)} \right],$$

or,

$$\gamma = -\frac{1}{2} \frac{\alpha}{Q_*} q_2, \quad (32)$$

where  $q_2 \equiv 3M - \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}$ .

Next we multiply Eq. (30) by  $\delta$  and Eq. (31) by  $\beta$ , and subtract the equations. To have the equations decoupled we now require that

$$2Q_*\delta^2 + 6M\beta\delta - 2Q_*(l-1)(l+2)\beta^2 = 0.$$

Solving this equation for  $\delta$ , and requiring that  $\delta$  will not vanish for  $l = 1$ , we find that

$$\delta = -\frac{1}{2}\frac{\beta}{Q_*} \left[ 3M + \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)} \right],$$

or,

$$\delta = -\frac{1}{2}\frac{\beta}{Q_*}q_1, \quad (33)$$

where  $q_1 \equiv 3M + \sqrt{9M^2 + 4Q_*^2(l-1)(l+2)}$ . The parameters  $\alpha$  and  $\beta$  remain to be fixed arbitrarily, although one could let them equal their dipole-case counterparts.

Substitution of the expressions for the decoupling parameters and the ansatz (28) and (29) in Eqs. (22) and (23) we find that the two decoupled wave equations assume the form

$$\Lambda^2 \bar{Z}_1^{(-)} = \frac{\Delta}{r^5} \left[ l(l+1)r + 4\frac{Q_*^2}{r} + \frac{q_2}{q_2 - q_1} (2q_1 - 6M) \right] \bar{Z}_1^{(-)}, \quad (34)$$

and

$$\Lambda^2 \bar{Z}_2^{(-)} = \frac{\Delta}{r^5} \left[ l(l+1)r + 4\frac{Q_*^2}{r} + \frac{q_1}{q_1 - q_2} (2q_2 - 6M) \right] \bar{Z}_2^{(-)}. \quad (35)$$

These equations can be put in a more compact and symmetrical form as

$$\Lambda^2 \bar{Z}_i^{(-)} = V_i^{(-)} \bar{Z}_i^{(-)}, \quad (36)$$

where the effective potential is given by

$$V_i^{(-)} = \frac{\Delta}{r^5} \left[ l(l+1)r + 4\frac{Q_*^2}{r} - q_j \right], \quad (37)$$

and  $i, j = 1, 2$ ,  $i \neq j$ . This wave equation is similar to the equation given in Ref. [5]. However, in our case this equation holds for all the multipole modes, including the dipole, whereas in Ref. [5] the dipole modes are excluded, because of the inapplicability of the perturbative formalism to the dipole modes.

## VI. THE COMPLETION OF THE SOLUTION

We can now complete the solution by giving the metric coefficients and the components of the Maxwell tensor in

terms of the functions  $\bar{Z}_i^{(-)}$  (for dipole modes just  $\bar{Z}_2^{(-)}$ ). Knowing the functions  $\bar{Z}_i^{(-)}$  the functions  $\bar{H}_i^{(-)}$  can be calculated using Eqs. (28) and (29). Then, the functions  $Q$  and  $B$  can be computed by Eqs. (20) and (21). Using Eq. (8) and the (linearized) Maxwell equations (5), (6), and (7) all the perturbed components of the Maxwell tensor are calculable. From Eq. (21) the function  $Q$  is known. Then, due to Eq. (9), we know the combination  $q_{2,3} - q_{3,2}$  of the perturbations of the metric coefficients. Using Eqs. (10) and (11) we can now obtain explicit expressions for the metric coefficients, and thus complete the solution.

We now have a perturbative formalism which is valid for the treatment of all the multipole modes, including the dipole, and which generalizes the formalism given in Ref. [5]. (The formalism for the polar modes is given in Paper I.) In a similar way to Ref. [5], one can now show the relations between the polar and the axial modes. (This is done most easily via the Newman-Penrose formalism—see Ref. [5].)

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